Online Directed Spanners and Steiner Forests

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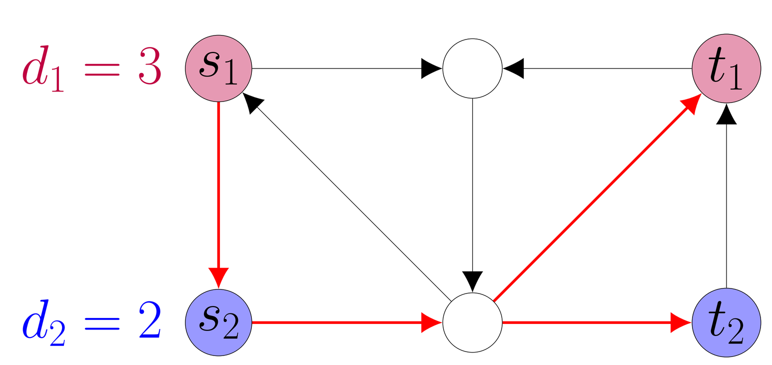
Directed Pairwise Spanners



- Each edge e has length $\ell_e \in \mathbb{R}_{\geq 0}$.
- Edge lengths are uniform if $\ell_e = 1$ for all e.

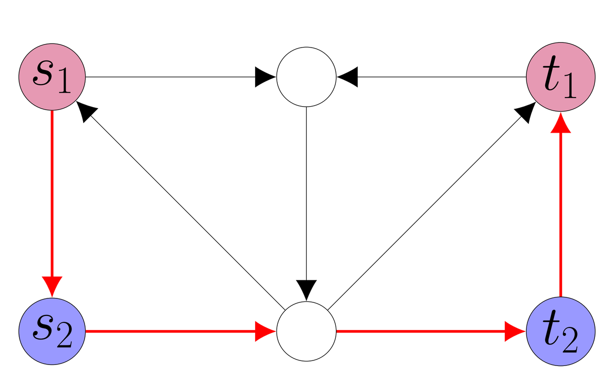
Goal: pick minimum #edges such that $d(s_i, t_i) \leq d_i$ for all i.

Online Directed Pairwise Spanners

- The graph is known offline.
- (s_i, t_i) and d_i arrive one at a time.

Goal: irrevocably pick minimum total #edges such that $d(s_i, t_i) \leq d_i$.

Directed Steiner Forests



- Each edge e has cost $c_e \in \mathbb{R}_{\geq 0}$.
- Edge costs are uniform if $c_e = 1$ for all e.

Goal: pick edges e so that

- the total cost $\sum_{e \text{ is picked}} c_e$ is minimized and
- there is an $s_i \rightsquigarrow t_i$ path for all i.

Online Directed Steiner Forests

- The graph is known offline.
- (s_i, t_i) arrives one at a time.

Goal: irrevocably pick edges e so that

- the total cost $\sum_{e \text{ is picked}} c_e$ is minimized and
- there is an $s_i \rightsquigarrow t_i$ path for all i.

Our Results (text in red and blue):

Setting	Pairwise Spanners	Directed Steiner Forests
Offline	$\tilde{O}(n^{3/5+\varepsilon})$ (uniform) [CDKL17']	$\tilde{O}(n^{26/45+\varepsilon})$ (uniform) [AB18']
		$O(n^{2/3+\varepsilon})$ [BBMRY13']
	$\tilde{O}(n^{4/5})$, $\tilde{O}(k^{1/2+\varepsilon})$ (uni.)	$O(k^{1/2+\varepsilon})$ [CEGS11']
Online	$\tilde{O}(n^{4/5})$	$\tilde{O}(k^{1/2+\varepsilon})$ [CEKP15']
	$\tilde{O}(n^{2/3+\varepsilon})$ (uniform)	$\tilde{O}(n^{2/3+\varepsilon})$ (uniform)
	$\tilde{O}(k^{1/2+\varepsilon})$ (uniform)	

n: number of vertices, k: number of terminal pairs

All our algorithms run in polynomial time. $\tilde{O}(k^{1/2+\varepsilon})$ (uniform) is deterministic while our other algorithms are randomized.

Main Tool: Online Covering

$$\text{minimize } \mathbf{c}^T x \text{ over } x \in \mathbb{R}^n_{\geq 0} \text{ s.t. } Ax \geq \mathbf{1}.$$

where $A \in \mathbb{R}^{m \times n}_{\geq 0}$ consists of m covering constraints and $\mathbf{c} \in \mathbb{R}^n_{>0}$.

In the online setting, the covering constraints arrive one at a time.

Goal: Irrevocably increase x such that all covering constraints are satisfied and the objective is approximately minimized.

This problem naturally extends to the setting where violating constraints are found by a separation oracle.

Theorem (Online Covering)

There exists an $O(\log n)$-competitive algorithm for online covering which encounters polynomially many violating constraints.

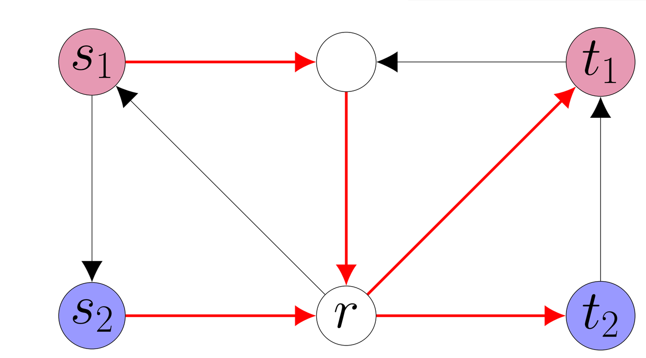
- Proof Sketch (Following Buchbinder and Naor 05'): Consider the dual packing problem. Increment the covering variables in exponential load w.r.t. the dual variables until the arriving covering constraint is satisfied by a factor of 2. Use the relation between the primal and dual solutions.

Our Main Theorem

For the online pairwise spanner problem with uniform edge lengths, there exists a deterministic polynomial time algorithm with competitive ratio $\tilde{O}(k^{1/2+\varepsilon})$ for any constant $\varepsilon > 0$.

Proof Sketch:

- Existence of an $O(\sqrt{k})$-approximate junction tree solution



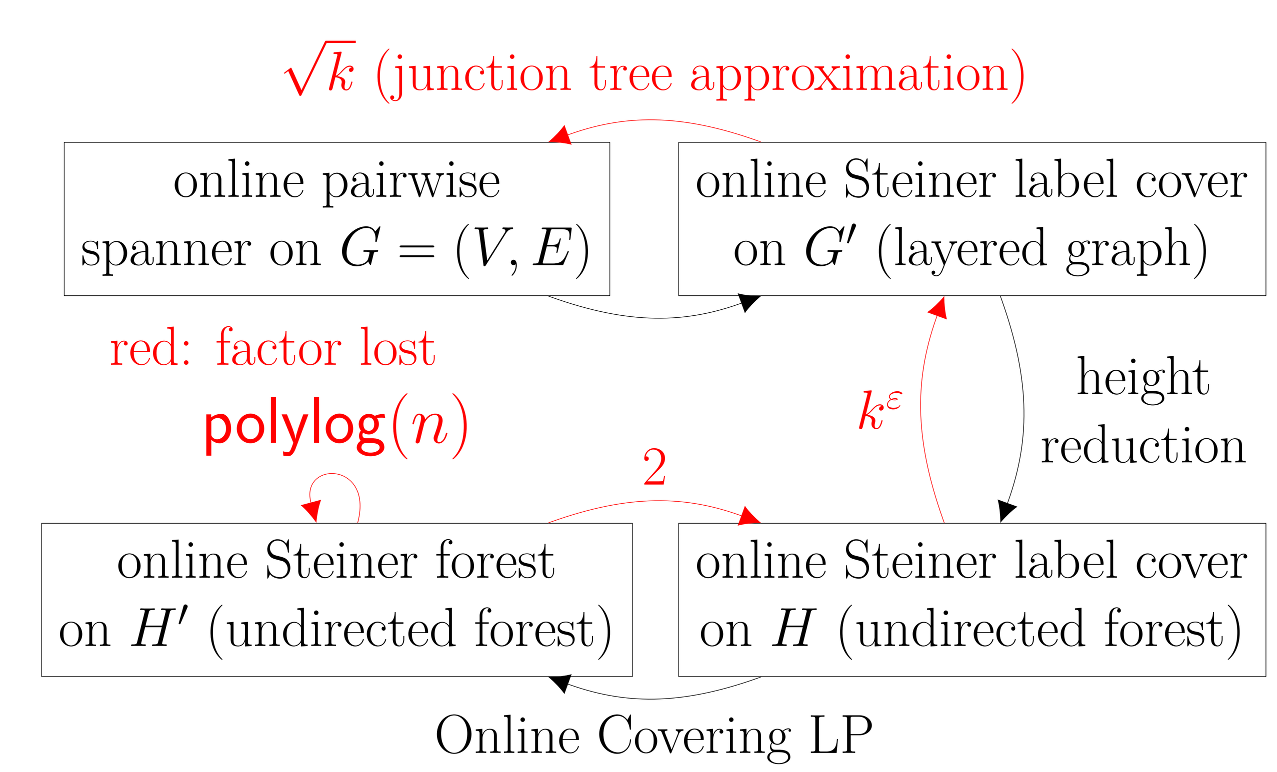
- Reduction from online directed spanner to online Steiner label cover and the height reduction technique

  Using ideas from [CDKL17'] (density → global approximation)

- Reduction online Steiner label cover to online undirected Steiner forest

  Using ideas from [CEKP15']

Proof Sketch by a Picture



Overall: $\tilde{O}(k^{1/2+\varepsilon})$-competitive