# Local Access To Random Walks [1]

Amartya Shankha Biswas (MIT), Edward Pyne (Harvard) and Ronitt Rubinfeld (MIT)

## Problem

Given a graph $G$ on $n$ vertices, take a random walk of length $L$. This requires time $\Omega(L)$. But what if we only want the position of the walk at a subset of those times?

Given $G$ and a sequence of queries $t_1, t_2, t_3, \ldots,$ return $v_1, v_2, \ldots \in V$ such that $v_i$ is the position of a random walk at time $t_i$, and the distribution over responses $\{v_i\}$ is close in $\ell_1$ distance to the projection of a truly random walk at those timesteps. We call this a **local access algorithm**.

## Results: Upper Bounds

Hide (polynomial) dependence on degree $d$.

1. For an undirected graph with spectral gap $\lambda$, algorithm with per-query runtime
$$\widetilde{O}\left(\frac{1}{\lambda}\sqrt{n}\right).$$

$\Longrightarrow$ For expander graphs, per-query runtime of $\widetilde{O}(\sqrt{n})$.

2. For Abelian Cayley graphs (hypercubes, cycles...), algorithm with per-query runtime $\widetilde{O}(\log n)$.

## Results: Lower Bounds

A local-access algorithm given random_neighbor probe access to **random regular graphs** must have runtime at least:

1. $\Omega(\sqrt{n}/\log n)$ per query if adversary is allowed to see probes.
2. $\Omega(n^{1/4})$ per query even against a fixed query sequence.

## Approach: $\widetilde{O}(\sqrt{n})$ Upper Bound

Suppose the spectral gap of $G$ is at least $1/20$. Then walks of length $M = O(\log n)$ from any vertex $v \in V$ are $1/\mathrm{poly}(n)$-close to the stationary distribution. For simplicity, assume $G$ is regular.

**Invariant:** Previously determined times are either adjacent or $\geq 2M$ steps apart. Given a new query at time $t$, let closest determined times be $t_- < t < t_+$.

1. If $|t - t_-| \geq 2M$ and $|t - t_+| \geq 2M$, return a random vertex.
2. If $|t - t_-| \leq 2M$ and $|t - t_+| \geq 2M$, return all positions $[t_-, t]$ by taking a random walk from $v_{t_-}$.
3. If $|t - t_-| \leq 2M$ and $|t - t_+| \leq 2M$, take $\sqrt{n}$ random walks from $t_-$ and $t_+$. If a collision occurs, stitch the colliding walks together and return.

## Key Tool: Edge Probabilities in $G(n, d)$

Let $S$ be a set of edges in a random $d$-regular graph, where $|S| = o(n)$. Let $\mathfrak{G}$ be the set of $d$-regular graphs containing $S$. Then for any $(u, v)$:

$$\Pr_{G \in \mathfrak{G}}[(u, v) \in G] \leq \frac{O_d(1)}{n}.$$

Our uses:

- Random walks of length $o(\sqrt{n})$ define trees with probability $1 - o(1)$.
- Any algorithm given probe access to $G(n, d)$ will fail to find a cycle with $o(\sqrt{n})$ probes.

Proof uses configuration model of Bollobas [2], strengthening due to [3].

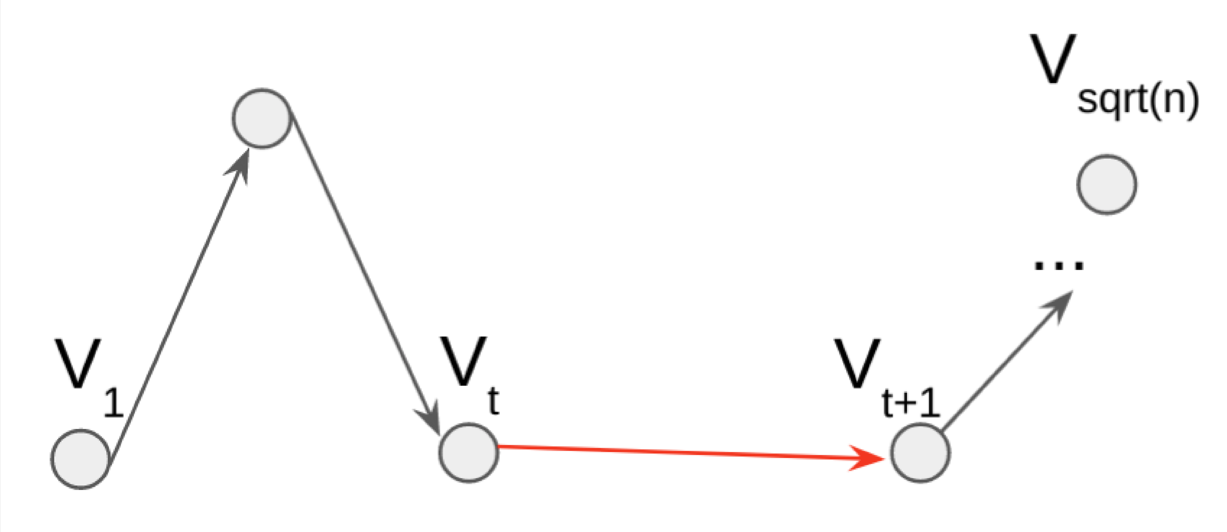
## Approach: $\widetilde{\Omega}(\sqrt{n})$ Lower Bound

Model: Local access algorithm with probe access to a random regular graph.

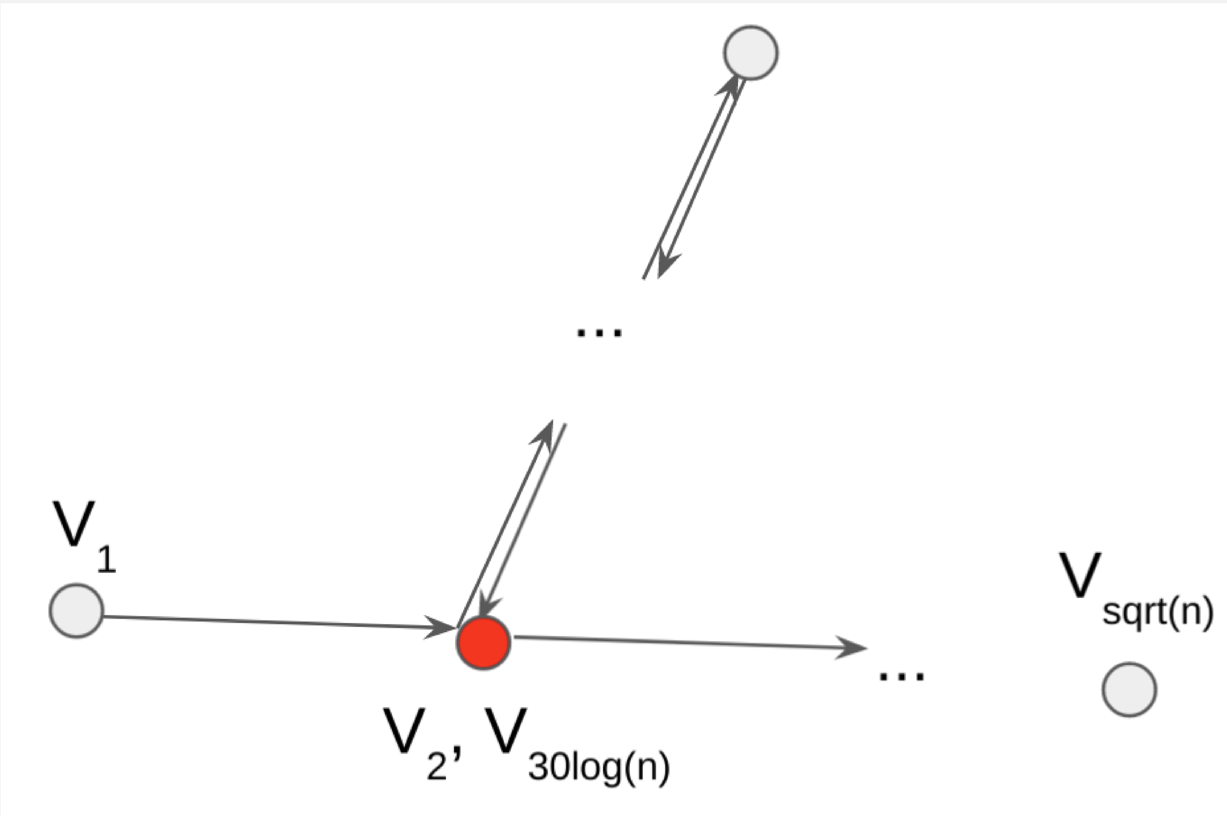
**Key Idea 1:** Any such algorithm cannot find a cycle in the graph with non-negligible probability with fewer than $O(\sqrt{n})$ probes.

**Key Idea 2:** If we first ask for the positions of the walk at time $1, \sqrt{n}$, the algorithm won't have time to take an honest walk from $v_1$ of length $\sqrt{n}$. Then make $O(\log n)$ further queries to narrow down the "bad" segment to length $O(\log n)$, and query all times in this segment. The returned walk satisfies one of two properties that distinguish it from truly random:

1. It "traverses" an edge that isn't actually present.




2. It revisits a vertex after $\Omega(\log n)$ steps.




## Further Results

Above lower bound requires knowing the probes the algorithm has made to the graph (all algorithms we give succeed even with this allowance). Obtain $n^{1/4}$ lower bound with query sequence fixed in advance.

For graphs with algebraic structure (Cartesian products, tensor products, Abelian Cayley) can obtain much faster $\mathrm{polylog}(n)$ runtime. In Abelian case, exploit that endpoint is invariant under changing order of edges to only sample *counts* of edge labels, not precise sequence.

## Further Questions

Local access to more objects - random tilings?
Local access to more graphs - non-Abelian Cayley graphs, other structured classes?

## References


[1] Amartya Shankha Biswas, Edward Pyne, and Ronitt Rubinfeld.
Local access to random walks.
*CoRR*, abs/2102.07740, 2021.

[2] Béla Bollobás.
A probabilistic proof of an asymptotic formula for the number of labelled regular graphs.
*European Journal of Combinatorics*, 1(4):311–316, 1980.

[3] Brendan D McKay and Nicholas C Wormald.
Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$.
*Combinatorica*, 11(4):369–382, 1991.